

Metric gravity theories and cosmology.

II. Stability of a ground state in $f(R)$ theories

Leszek M. SOKOŁOWSKI

Astronomical Observatory and Centre for Astrophysics,
Jagellonian University,
Orla 171, Kraków 30-244, Poland
uflsokol@th.if.uj.edu.pl

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Abstract

In the second part of investigation of metric nonlinear gravity theories we study a fundamental criterion of viability of any gravity theory: the existence of a stable ground state solution being either Minkowski, de Sitter or anti-de Sitter space. Stability of the ground state is independent of which frame is physical. In general a given theory has multiple ground states and splits into independent physical sectors. The fact that all $L = f(g_{\alpha\beta}, R_{\mu\nu})$ gravity theories (except some singular cases) are dynamically equivalent to Einstein gravity plus a massive spin-two and a massive scalar field allows to investigate the stability problem using methods developed in general relativity. These methods can be directly applied to $L = f(R)$ theories wherein the spin-two field is absent. Furthermore for these theories which have anti-de Sitter space as the ground state we prove a positive-energy theorem allowing to define the notion of conserved total gravitational energy in Jordan frame (i.e., for the fourth-order equations of motion). As is shown in thirteen examples of specific Lagrangians the stability criterion works effectively without long computations whenever the curvature of the ground state is determined. An infinite number of gravity theories have a stable ground state and further viability criteria are necessary.

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1 Introduction

In our previous work [1] (hereafter cited as Paper I) we have argued that the cosmological observations based on the spatially flat Robertson–Walker spacetime (usually fitted

by the Λ CDM model) are unsuitable and insufficient to reconstruct the Lagrangian of the true gravity theory which correctly accounts for the present epoch of cosmic acceleration. The very fact that the observational data may be fitted by a huge collection of diverse Lagrangians clearly indicates that the idea of reconstructing the correct theory from cosmology is implausible and a deeper investigation confirms that the task is impossible. If one believes that general relativity needs some modifications, these should not be directly induced from the approximate data. Rather, as any other new physical theory, a new gravity theory should be based on new concepts and ideas. In the present case the modifications consist in replacing the Einstein Lagrangian $L = R$ by a nonlinear function of Riemann tensor and it is difficult to give a physical idea which would uniquely choose the correct function. One should rather apply various criteria taken from classical field theory to maximally narrow down the class of viable theories. In our opinion the most fundamental condition is the existence of a maximally symmetric stable ground state solution to the equations of motion in pure gravity. Ordinary matter cannot destroy stability of a ground state if it is stable against purely gravitational perturbations. Stability of the ground state is independent of which frame (the set of dynamical field variables) is regarded as physical. Hence the stability can be investigated in Einstein frame where the most reliable methods of checking it have been developed.

In the case of $L = f(R)$ the stability is verified in this frame in a very effective and quick way. The main step is to solve an algebraic equation to determine the curvature R of a ground state. If the equation is solvable for a given Lagrangian one easily establishes whether the state is stable. It is worth noting that our method based on the dominant energy condition for the scalar component of gravity, applies to generic (inhomogeneous) perturbations of a ground state, while most authors assume homogeneous or at most spherically symmetric perturbations.

We emphasize that stability of the ground state and other possible viability conditions concern physical viability of a gravity theory from the viewpoint of field theory, i.e. they concern its internal structure and its relationships to other physical theories. There are many viable gravity theories. At the present level of knowledge there is no system of selection rules (i.e. viability conditions) allowing one to uniquely determine the correct theory. Hence physical viability has a restricted meaning and does not mean that a viable theory necessarily fits in a satisfactory way some empirical data. Actually most of viable gravity theories is in conflict with observations. In particular the physical viability (which in the present work coincides with the stability of the ground state) is independent of a cosmological viability introduced by Amendola et al. [2]. According to these authors the cosmological viability means a satisfactory evolution of the universe in the flat ($k = 0$) Robertson–Walker spacetime: long matter era with the cosmic scale factor $a \propto t^{2/3}$ prior to a late-time acceleration epoch.

The present work deals with various aspects of the stability problem: determination of ground state solutions and their multiplicity, notion of stability, total energy and its relationship to stability of a ground state, reliability of various methods of checking the stability. Finally we formulate a stability condition in terms of a potential in Einstein frame for the scalar component of a $L = f(R)$ gravity. We apply the condition to thirteen specific Lagrangians, mainly taken from the literature and show¹ that some theories which are cosmologically viable are physically untenable.

2 Stability of a maximally symmetric ground state

A minimal requirement that may be imposed on a gravity theory to be viable is that it has a classically stable maximally symmetric ground state solution. In some classical field theories, e.g. in Liouville field theory [4] a ground state may not exist, but in gravitational physics the existence of a ground state hardly needs justification. In a metric gravity theory gravitational interactions are manifested by the dynamical curvature of the spacetime, hence in the absence of these interactions the spacetime should be either flat or maximally symmetric with the nongeometric components of the gravitational multiplet equal to zero or covariantly constant. Therefore the spacetime of the ground state for any NLG theory may be Minkowski, de Sitter (dS) or anti-de Sitter² (AdS) space. For simplicity we assume spacetime dimensionality $d = 4$ although our arguments (with slight modifications) will also hold in $d > 4$. Classical stability means that the ground state solution is stable against small excitations of the (multicomponent) gravitational field and small excitations of a given kind of matter sources, i.e. there are no growing in time perturbation modes. In principle a viable classical field theory may admit a semiclassical instability: the ground state is separated by a finite barrier from a more stable (in the sense of lower energy) state and can decay into it by a semiclassical barrier penetration [5]. We shall not consider this possibility and focus our attention on classical stability, hereafter named stability.

A question that may arise at the very beginning of investigation of the problem is whether a metric NLG theory, being a higher derivative one, can at all be stable [6]. In point particle mechanics one may invoke to this end the old famous Ostrogradski theorem to the effect that if a mechanical Lagrangian depends on second and higher time derivatives of the particle positions (which cannot be eliminated by partial integration) the corresponding Hamiltonian is linear in at least one canonical momentum and thus is unbounded from below. As a consequence there are both positive and negative energy

¹The problem of the cosmic acceleration applying $f(R)$ gravity was also investigated within the Kaluza–Klein framework [3] and some results achieved there are akin to ours.

²By anti-de Sitter space we always mean the covering anti-de Sitter space without closed timelike curves.

states and if the particles are interacting the theory is unstable since any solution decays explosively due to self-excitation: unlimited amount of energy is transferred from negative energy particles to positive energy ones. By analogy, the same (or rather more drastic and violent) instability is expected to occur in classical (and quantum) field theory with higher time derivatives. Thus a generic NLG theory should be inherently unstable and hence unphysical. We admit that the problem is important and deserves a detailed investigation. Here we wish only to make short comment on how it is possible to avoid this conclusion.

We stress that the Ostrogradski theorem is a rigorous "no-go theorem" in classical and quantum point particle mechanics [6] while in metric NLG theories it may only be conjectured by analogy. In fact, a mechanical Hamiltonian determines energy and if it is indefinite (and unbounded from below) it signals that self-excitation processes are likely to occur. Recall that a metric theory of gravity is based on the equivalence principle what implies that the notion of gravitational energy density makes no sense. Yet in a field theory in Minkowski space the field energy density is equal to the Hamiltonian density and the latter is (for known fields) positive definite. In the canonical ADM formalism in general relativity the canonical momenta are defined in an intricate way (including constraints), not akin to that in point mechanics and the total ADM energy is to large extent independent of the detailed form of the Hamiltonian density (which is indefinite). Therefore in general relativity the relationship between stability (understood as the positivity of energy, see below) and the form of the Hamiltonian density is very indirect, practically broken. In metric NLG theories the Legendre transformations from Jordan frame to Helmholtz-Jordan frame (HJF) and Einstein frame (EF) map the higher derivative theory to Einstein gravity plus nongeometric components of the multiplet which dynamically act as some matter fields, therefore the stability problem in these theories is reduced to that in the latter theory. The Ostrogradski theorem may rather serve as a warning that some troubles may appear there and in fact troubles were found (the ghost-like behaviour of the massive spin-2 component of gravity) without resorting to it. Note that the notion of "inherently unstable theory" is imprecise: stability always concerns a given solution. And what is really required from a viable gravity theory is existence of a stable ground state solution; stability of excited states is a different problem.

In the physical literature there is some confusion concerning stability since there are actually two notions of stability: dynamical stability (stability of evolution) meaning that there are no growing modes and stability as a consequence of positivity of total energy. It has been believed for a long time that the two notions are identical and since investigations of energy are relatively easier the research was first centered on it. Stability in the context of energy was developed in a series of papers which will be referred here to as "classical works". Positivity of the ADM energy implies stability of Minkowski space. The notion of this energy was then extended to the Abbott-Deser (AD) energy for the spacetimes which are asymptotically de Sitter or anti-de Sitter [7].

Applying this notion it was shown that vacuum dS is linearly stable [7] while AdS space is both linearly and nonlinearly stable in vacuum [7] and in presence of any matter satisfying the dominant energy condition (DEC) in any dimension $d \geq 4$ [8, 9]. However it was found that stability does not necessarily result from the positivity of energy: there are situations in which the positive energy theorem holds and instabilities develop [10]. Thus dynamical stability (no growing in time perturbation modes) and positivity of energy are quite different unrelated things. Stability of evolution requires mathematically rigorous investigations.

In the rigorous approach it was shown that Minkowski space is globally dynamically stable: in vacuum [11], in presence of the electromagnetic field [12] or of the linear massless scalar field [13]. Vacuum de Sitter space is globally stable in four [14] and any larger even number of dimensions [15]. Inclusion of matter is difficult: global stability of dS space was proved only in the case of Yang–Mills fields (in $d = 4$) [16] and for a scalar field with a very specific potential [17]; its stability for all other forms of matter is unknown. Even less is rigorously known about stability of anti-de Sitter space: it is globally linearization stable for the Maxwell and linear scalar field [18] and for the vacuum case Friedrich [19] proved finite time nonlinear stability. There are no rigorous global results, it is only believed that vacuum AdS space is dynamically stable and nothing has been investigated in the case of self-interacting scalar fields.

While the fully reliable rigorous results are quite modest from the standpoint of a physicist dealing with gravitational fields generated by a rich variety of matter sources, the classical theorems based on the positivity of energy are, from the viewpoint of mathematicians, of rather little reliability [20]. In proving the rigorous theorems only the exact field equations are relevant and the dominant energy condition does not play explicitly any role. However in the few cases where matter sources are present, DEC does hold. It is therefore reasonable to conjecture that Minkowski, de Sitter and anti-de Sitter spaces are globally nonlinearly stable only if any self-gravitating matter does satisfy the condition. The conjecture is supported by outcomes found in the linear approximation to semiclassical general relativity where the expectation value $\langle 0|T_{\mu\nu}|0 \rangle$ cannot satisfy DEC due to the particle creation by the gravitational field. In the presence of the electromagnetic, neutrino and massless scalar fields Minkowski space is linearly unstable [21] and similarly a minimally coupled quantum scalar field renders Sitter space linearly unstable [22].

All the aforementioned papers deal with solutions to Einstein field equations. Recently Faraoni [23] studied stability of vacuum dS space in restricted NLG theories in Jordan frame for the fourth-order field equations. The dS metric can be presented in the form of the spatially flat Robertson–Walker spacetime and he has applied the gauge invariant formalism of Bardeen–Ellis–Bruni–Hwang for perturbations of Friedmann cos-

mology. The formalism works for any field equations in this background and he proves linearization stability of dS space: scalar and tensor metric perturbations are fading or oscillating at late times provided the Lagrangian $L = f(R)$ satisfies some inequality. In this formalism the physical meaning of this crucial inequality is unclear. It turns out that the condition is equivalent to the condition that the (positive) potential for the scalar component of gravity in Einstein frame attains minimum at dS space being a ground state solution, see sect. 6. The BEBH formalism does not apply to perturbations of AdS space since its metric cannot be expressed as the spatially flat R–W spacetime. It is interesting to see that in most papers on NLG theories it is assumed that a curved ground state is necessarily dS space while AdS space is omitted without mention³.

We shall investigate stability of the maximally symmetric ground state solutions in various NLG theories in a coordinate independent manner. We presume that the classical works provide the correct assumptions under which the dynamical stability of these solutions will be rigorously proved in future. We shall work in Einstein frame where the only source for the metric is the scalar field component of gravity since on physical grounds it is stability of pure gravity that is crucial. Moreover we argue in section 4 that inclusion of matter (e.g. perfect fluid) does not affect stability of the solutions. We emphasize that stability of a candidate ground state solution is independent of which frame is regarded as physical since boundedness of solutions remains unaltered under Legendre transformations. The method based on positivity of total ADM or AD energy works directly only in Einstein frame. The energy–momentum tensor of the scalar satisfies the dominant energy condition if and only if its potential is nonnegative. Thus satisfying DEC for the field becomes an effective viability criterion for restricted NLG theories.

3 Candidate ground state solutions

We shall now investigate existence of candidate ground state (CGS) solutions, i.e., maximally symmetric (dS, AdS or Minkowski space) solutions in a restricted NLG theory with $L = f(R)$ for arbitrary f . A CGS solution becomes a true physical ground state solution (named vacuum) if it is stable. We assume that the Lagrangian has the same dimension as the curvature scalar, $[f(R)] = [R] = (\text{length})^{-2}$, and the signature is $(-+++)$. The field equations in Jordan frame are

$$E_{\mu\nu}(g) \equiv f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \nabla_\mu \nabla_\nu f'(R) + g_{\mu\nu} \square f'(R) = 0 \quad (1)$$

or

$$f'(R)R_{\mu\nu} - \nabla_\mu \nabla_\nu f'(R) + \frac{1}{3}g_{\mu\nu} \left[\frac{1}{2}f(R) - Rf'(R) \right] = 0,$$

³AdS space is mentioned as a possible ground state e.g. in [24, 25].

here $f' \equiv \frac{df}{dR}$ and $\square \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu$. In general $f(R)$ cannot be everywhere smooth and the nonlinear equations (1) require f be piecewise of C^3 class⁴.

A CGS solution exists if and only if the field equations (1) admit a class of Einstein spaces, $R_{\mu\nu} = \frac{1}{4}\lambda g_{\mu\nu}$, for some curvature scalar $R = \lambda$, as solutions. Since $\lambda = \text{const}$ and assuming that λ lies in the interval where $f(\lambda)$, $f'(\lambda)$, $f''(\lambda)$ and $f'''(\lambda)$ are finite, eqs. (1) reduce to an algebraic equation

$$\lambda f'(\lambda) - 2f(\lambda) = 0. \quad (2)$$

This equation was first found by Barrow and Ottewill [26] and then rediscovered many times. In general this equation has many solutions and to each solution $\lambda = \lambda_i$ there corresponds a whole class of Einstein spaces containing a maximally symmetric space-time, being dS for $\lambda_i > 0$, AdS for $\lambda_i < 0$ or Minkowski space (\mathcal{M}) for $\lambda_i = 0$. For some λ_i the maximally symmetric space may be stable. Each stable ground state (vacuum) defines a separate dynamical sector of the theory. Multiplicity of vacua for a $L = f(R)$ gravity was first noticed in [27].

We view (2) as an algebraic equation and assume that it has at most the countable number of solutions. Not every function $f(R)$ admits a solution to (2). First note the degenerate case where any value of λ is a solution (uncountable number of solutions); this occurs when (2) is viewed as a differential equation for f . Then $f(R) = aR^2$ for any constant $a \neq 0$ [26]. In the following we make some comments on this degenerate case. The eq. (2) has no solutions if its LHS defines a function of R , $Rf'(R) - 2f(R) \equiv F(R)$, which nowhere vanishes. Treating this definition as a differential equation for f (for a given F) one finds that any Lagrangian which admits no CGS solutions is of the form

$$f(R) = aR^2 \pm R^2 \int \frac{F(R)}{R^3} dR \quad (3)$$

with arbitrary $F(R) > 0$ everywhere. Examples.

1. Let $F(R) = Ce^{bR}$ with $b, C \neq 0$. Then

$$f(R) = aR^2 + C \left[-\frac{1}{2}e^{bR}(1 + bR) + \frac{b^2}{2}R^2 \int \frac{e^{bR}}{R} dR \right],$$

the latter integral is non-elementary.

2. For

$$F(R) = c_0 + \sum_{n=1}^{\infty} c_{2n} R^{2n}$$

⁴Any Lagrangian is determined up to a divergence of a vector field made up of the dynamical variables. If the gravitational Lagrangian is to be a scalar function of the Riemann tensor invariants alone and involve no derivatives of the curvature, the Lagrangian is determined up to a constant multiplicative factor. The factor must be fixed if any matter is minimally coupled to gravity in JF.

with c_0 and $c_{2n} > 0$ one gets

$$f(R) = -\frac{c_0}{2} + aR^2 + c_2R^2 \ln |R| + \frac{1}{2} \sum_{n=2}^{\infty} \frac{c_{2n}}{n-1} R^{2n}.$$

Any gravity theory with a Lagrangian of the form (3) is unphysical and should be rejected. Clearly there are infinitely many functions $f(R)$ admitting solutions to (2) and thus possibly possessing a stable vacuum. A few examples of these Lagrangians.

1. $f(R) = R + aR^2 + \alpha^{-2}R^3$, $\alpha > 0$. There are three CGS solutions: \mathcal{M} ($\lambda_1 = 0$), dS ($\lambda_2 = +\alpha$) and AdS ($\lambda_3 = -\alpha$).
2. $f(R) = -2\Lambda + R + aR^2 + bR^3$.
Eq. (2) becomes a cubic equation $\lambda^3 - \lambda/b + 4\Lambda/b = 0$ having 3 distinct real roots if $4\Lambda^2 < (27b)^{-1}$ and one real root λ_1 (and $\lambda_2 = \bar{\lambda}_3$ complex) if $4\Lambda^2 > (27b)^{-1}$. In the limiting case $4\Lambda^2 = (27b)^{-1}$ there are 3 real roots with $\lambda_1 = \lambda_2$ and $\lambda_3 \neq \lambda_1$. The solutions are independent of the coefficient a . In the case $b = 0$ there is a unique CGS solution $\lambda = 4\Lambda$, being dS or AdS, the same as in Einstein theory.
3. For $f(R) = \frac{1}{a}e^{aR}$, $a > 0$, there is a dS space with $\lambda = 2/a$ as a unique CGS solution.
4. For $f(R) = a\sqrt{R} + bR^{7/3}$, $a, b > 0$, there are two CGS solutions: $\lambda_1 = 0$ and $\lambda_2 = +(\frac{9a}{2b})^{6/11}$.

The last example illustrates a general rule: if $f(0) = 0$ then $\lambda = 0$ is always a solution of (2), i.e., \mathcal{M} is a CGS solution even if $f'(0)$ is divergent. In fact, if $f'(0) \rightarrow \pm\infty$ the term $Rf'(R)$ may a priori either vanish at $R = 0$ or diverge either logarithmically or as inverse of a power law. If $Rf'(R) = R^{-n}$ for some $n > 0$ then $f = c - \frac{1}{n}R^{-n}$, while for $Rf'(R) = (\ln R)^n$, $n > 0$, one finds $f = c + \frac{1}{n+1}(\ln R)^{n+1}$; in both cases $f(0)$ is divergent. For $f(0) = 0$ and $f'(0) \rightarrow \pm\infty$ the leading term in f near $R = 0$ is R^a , $0 < a < 1$, and then $Rf'(R) \rightarrow 0$. However if $f'(0)$ and/or $f''(0)$ is divergent the method for establishing whether \mathcal{M} is stable does not work.

We emphasize that in order to investigate the dynamics of a restricted NLG theory one needs *exact* solutions of eq. (2). We shall see that stability of a CGS solution is determined by the values of $f'(\lambda)$ and $f''(\lambda)$. In principle to check stability it is sufficient to find numerically an approximate solution λ to eq. (2) and then approximate values of $f'(\lambda)$ and $f''(\lambda)$. Also the mass of the scalar component of gravity is determined by these two numbers. However an exact solution is necessary to calculate the scalar field potential both in Helmholtz–Jordan and Einstein frames; otherwise one gets only approximate equations of motion in these frames as is shown in the following example: $L = f(R) = a \sin \frac{a}{R}$, $a > 0$. Introducing a dimensionless quantity $x \equiv \frac{a}{\lambda}$ one finds that eq. (2) reads $x \cos x + 2 \sin x = 0$. $\cos x = 0$ is not a solution and the equation may be written as $x + 2 \tan x = 0$. The obvious root is $x = 0$, but it corresponds to

$R = \lambda = \infty$ and this solution must be rejected on physical grounds. In the interval $-\pi/2 < x < \pi/2$ where tangens is continuous the functions x and $\text{tg } x$ are of the same sign and the equation has no solutions. In each interval $(n - 1/2)\pi < x < (n + 1/2)\pi$, $n = \pm 1, \pm 2, \dots$, the equation has exactly one solution which may be determined numerically. The scalar component of gravity is defined as $p = df/dR$ and to determine the potential for p one needs to invert this relation to get $R = r(p)$. In the present example $p = -(a/R)^2 \cos a/R$ and though this relation is in principle invertible (since $f''(R) \neq 0$ and f'' vanishes only at separate points where $\text{tg } a/R = 2R/a$), it cannot be inverted analytically in any of the intervals. One sees that exact solvability of eq. (2) is often correlated to exact invertibility of the definition $p = f'(R)$. We conclude that the condition of exact analytic solvability of eq. (2) is of crucial importance and in practice imposes stringent restrictions on the Lagrangians excluding many simple combinations of elementary functions. A further constraint will be imposed in the next section.

Finally we make two remarks on the field equations (1).

Firstly, recall that for cosmologists the most attractive Lagrangians are those containing inverse powers of R rather than being polynomials in R . In consequence the coefficients of fourth order derivatives in (1) are rational functions and this implies that one should deal with great care with various terms in these equations in order to avoid multiplying or dividing by zero⁵. For simplicity we demonstrate it on a toy model. Suppose that field equations read

$$R_{\mu\nu} + \frac{1}{R^2} \square R_{\mu\nu} = 0. \quad (4)$$

Multiplying them by R^2 one gets

$$R^2 R_{\mu\nu} + \square R_{\mu\nu} = 0 \quad (5)$$

and a class of solutions to these equations is given by $R_{\mu\nu} = \psi_{\mu\nu} \neq 0$ where the tensor is traceless, $R = \psi \equiv g^{\mu\nu} \psi_{\mu\nu} = 0$ and satisfies $\square \psi_{\mu\nu} = 0$. However $\psi_{\mu\nu}$ is not a solution to (4) since the LHS of these equations is then $\psi_{\mu\nu} + 0/0$. A class of solutions to (4) is of the form $R_{\mu\nu} = \phi_{\mu\nu} \neq 0$ and $\square \phi_{\mu\nu} = -\phi^2 \phi_{\mu\nu}$ with $\phi \equiv g^{\mu\nu} \phi_{\mu\nu}$; clearly these are also solutions to (5). Furthermore, any spacetime satisfying $R_{\mu\nu} = 0$ is a solution to both (4) and (5). At first sight this is not since the second term in (4) becomes divergent. One may however give a precise meaning to this term by trying an Einstein space,

⁵We stress that this is not trivial. In a frequently quoted paper [28] the trace of eqs. (1) for a Lagrangian $R - 1/R$ was multiplied by R^3 giving rise to a scalar equation for R admitting $R = 0$ as a solution and thus Minkowski space; further considerations of the work were based on perturbations of this spacetime. Actually the field equations for this Lagrangian have only dS and AdS spaces as CGS solutions. This "curvature instability" found in [28] has been generalized to many other functions $f(R)$ without checking if Minkowski space is a solution and is even regarded as advantage of the Palatini formalism over the purely metric gravity theories [29]. This error of introducing or omitting some classes of solutions by multiplying the field equations by a power of R may be traced back to Bicknell [30].

$R_{\mu\nu} = (\lambda/4)g_{\mu\nu}$, then $\square R_{\mu\nu} \equiv 0$ and the eqs. (4) reduce to $\lambda g_{\mu\nu} = 0$ so that $R_{\mu\nu} = 0$ actually are solutions. In conclusion, by replacing the correct equations (4) by allegedly equivalent equations (5) one introduces a class of false solutions $R_{\mu\nu} = \psi_{\mu\nu}$.

Secondly, we comment on the cosmological constant [31]. In metric NLG theories this notion has a rather limited sense. In general relativity Λ is both the constant appearing in the Einstein–Hilbert Lagrangian, $\Lambda = -\frac{1}{2}L(0)$, and the curvature of the unique maximally symmetric ground state, $\Lambda = R/4$. If $f(0) \neq 0$ is finite in an NLG theory one may define Λ as $-\frac{1}{2}f(0)$, however there is at least one CGS solution with the curvature $R = \lambda \neq 0$ whose value is independent of the value $f(0)$ (in the sense that the function $F(R) \equiv Rf'(R) - 2f(R)$ may be freely varied near $R = 0$ provided $F(0) \neq 0$ is preserved, then $R = \lambda$ remains the solution of (2)). Alternatively, Λ may be defined as $\lambda/4$ for each vacuum (stable ground state), then Λ has different values in different sectors of the theory. However this cosmological constant is related solely to the vacuum and does *not* appear as a parameter in other solutions to the field equations (1). We therefore shall not use this notion.

4 The field equations including matter

We shall now express the field equations in the form appropriate for investigating stability of the CGS solutions. Detailed calculations based on the general formalism [32, 33] are given in [34]. The scalar component of the gravitational doublet is defined in HJF as $p \equiv \frac{df}{dR}$, this canonical momentum is dimensionless. The definition is inverted to give the curvature scalar R as a function of p , $R(g) = r(p)$, i.e.,

$$f'(R)|_{R=r(p)} \equiv p.$$

The inverse function $r(p)$ exists iff $f''(R) \neq 0$. The pure gravity Helmholtz action

$$S_{HJ} = \int d^4x \sqrt{-g} L_H(g, p)$$

with $L_H = p[R(g) - r(p)] + f(r(p))$ (see Paper I) gives rise to the field equations

$$G_{\mu\nu}(g) = \theta_{\mu\nu}(p, g) \equiv \frac{1}{p} \nabla_\mu \nabla_\nu p - \frac{1}{6} \left[\frac{1}{p} f(r(p)) + r(p) \right] g_{\mu\nu} \quad (6)$$

and

$$\square p - \frac{2}{3} f(r(p)) + \frac{1}{3} p r(p) = 0. \quad (7)$$

By taking trace of (6) and employing (7) one recovers the relation $R(g) = r(p)$. The effective energy–momentum tensor for p contains a linear term signalling that the energy

density is indefinite and deceptively suggesting that all solutions, including the CGS ones, are unstable [6]. However $\theta_{\mu\nu}$ turns out unreliable in this respect and to study stability one makes the transformation from HJF to Einstein frame being a mere change of the dynamical variables. It consists of a conformal map of the metric,

$$g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} \equiv p g_{\mu\nu},$$

and a redefinition of the scalar,

$$p \equiv \exp\left(\sqrt{\frac{2}{3}}\kappa\phi\right)$$

or $\phi = \sqrt{\frac{3}{2}}\frac{1}{\kappa} \ln p$, with κ being a dimensional constant to be specified later⁶. Under the transformation of the variables the action integrals in HJF and EF are equal,

$$S_{HJ} = S_E = \int d^4x \sqrt{-\tilde{g}} \tilde{L}_H(\tilde{g}, p(\phi)), \quad (8)$$

what defines \tilde{L}_H . To get the total Lagrangian precisely as in general relativity one introduces an equivalent Lagrangian proportional to \tilde{L}_H ,

$$L_E \equiv \frac{1}{2\kappa^2 c} \tilde{L}_H = \frac{1}{2\kappa^2 c} \tilde{R}(\tilde{g}) + \frac{1}{c} L_\phi \quad (9)$$

and sets $(2\kappa^2 c)^{-1} \equiv c^3/(16\pi G)$ or $\kappa^2 = 8\pi G/c^4$. Hence ϕ is a minimally coupled scalar field with a self-interaction potential,

$$L_\phi = -\frac{1}{2}\tilde{g}^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} - \frac{1}{2\kappa^2} \left[\frac{r(p)}{p} - \frac{f(r(p))}{p^2} \right] \equiv -\frac{1}{2}\tilde{g}^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} - V(p(\phi)). \quad (10)$$

The constant κ determines the dimension of ϕ , $[\phi] = \text{g}^{1/2}\text{cm}^{1/2}\text{s}^{-1}$, while V acquires dimensionality of energy density. The field equations following from (9) are

$$\tilde{G}_{\mu\nu}(\tilde{g}) = \kappa^2 T_{\mu\nu}(\phi, \tilde{g}) = \kappa^2 (\phi_{,\mu}\phi_{,\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{g}^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} - \tilde{g}_{\mu\nu}V(\phi)) \quad (11)$$

and

$$\tilde{\square}\phi = \frac{dV}{d\phi} = \sqrt{\frac{2}{3}}\kappa p \frac{dV}{dp}. \quad (12)$$

Solutions for a self-interacting scalar field in general relativity were studied in many papers, however they are not solutions to eqs. (10)–(12) since the potential (10) is in most cases different from the potentials appearing in those papers. For example, an exponential potential $V_0 \exp(-\alpha\kappa\phi)$ with constant α was investigated in a number of works (see

⁶In Paper I, for simplicity we have put $\kappa = 1$ in eqs. (8), (9) of that paper and the definition of ϕ .

e.g. [35]); in terms of the scalar p it reads $V_0 p^{-\alpha}$, but there are no simple Lagrangians $L = f(R)$ generating this potential via eq. (10). Recall that as long as one considers pure gravity, i.e. there is no minimally coupled matter in JF, the original Lagrangian $L = f(R)$ is determined up to an arbitrary constant factor A . Let $\bar{f}(R) \equiv Af(R)$. Then $\bar{p} \equiv \bar{f}'(R) = Ap$ and the inverse relation is $R(g) = \bar{r}(\bar{p})$. On the other hand $R(g) = r(p)$ so that $\bar{r}(\bar{p}) = r(p) = r(\bar{p}/A)$. This implies $L_H(g, \bar{p}) = AL_H(g, p)$, the conformal factor \bar{p} generates in EF the metric $\hat{g}_{\mu\nu} = A\tilde{g}_{\mu\nu}$ and $\tilde{L}_H(\hat{g}, \bar{p}) = A^{-1}\tilde{L}_H(\tilde{g}, p)$.

The conformal map should not alter the signature of the metric, thus one requires $p > 0$. In general $f'(R)$ cannot be positive for all R and it is sufficient to require that the map preserve the signature at the CGS solutions, i.e., $p(\lambda) = f'(\lambda) > 0$ for each solution of eq. (2). Then $p > 0$ in some neighbourhood of $R = \lambda$. If $p(\lambda) < 0$ one should take the Lagrangian $L = -f(R)$. It may occur for some $f(R)$ having multiple solutions of (2) that $p(\lambda_i) > 0$ and $p(\lambda_j) < 0$ for $i \neq j$, then one should appropriately choose the sign of L at each sector of the theory separately. We shall assume that this has been done⁷ and $p(\lambda_i) = f'(\lambda_i) > 0$.

The transformation from HJF to Einstein frame exists in a neighbourhood of a CGS solution with $R = \lambda$ iff $f'(\lambda) \neq 0$. If $f'(\lambda) = 0$ the EF does not exist and the method of checking stability of the CGS solution does not apply. From $\lambda f'(\lambda) - 2f(\lambda) = 0$ it follows that $f(\lambda) = 0$ and assuming that f is analytic around $R = \lambda$ it has a general form

$$f(R) = \sum_{n=2}^{\infty} a_n (R - \lambda)^n \quad (13)$$

for any real λ . Notice that the degenerate Lagrangian $L = R^2$ belongs to this class. This class of singular Lagrangians needs a separate treatment (see section 6) and we assume that $f(R)$ is not of the form (13).

For Lagrangians which are different from (13) the potential $V(\phi)$ in EF is not a constant. To prove it one assumes that $V = \text{const}$ and determines the corresponding $f(R)$. From (10) one gets

$$r(p) = Cp + \frac{f(r)}{p} \quad (14)$$

where $C \equiv 2\kappa^2 V$ and one differentiates this equation with respect to f employing

$$\frac{dr}{df} = \left(\frac{df}{dr} \right)^{-1} = \frac{1}{p}.$$

One finds

$$\frac{dr}{df} = \frac{1}{p} = C \frac{dp}{df} + \frac{1}{p} - \frac{f}{p^2} \frac{dp}{df} \quad \text{or} \quad \frac{dp}{df} \left(C - \frac{f}{p^2} \right) = 0.$$

⁷One may try a simplification by choosing $L(R) = \frac{f(R)}{f'(R)}$, then $L'(\lambda) = 1$. Actually this choice does not simplify the expressions for derivatives of the potential V and we shall not apply it.

Since $\frac{dp}{df} = \left(\frac{df}{dr} \frac{dr}{dp}\right)^{-1} \neq 0$ this yields

$$f(r(p)) = Cp^2.$$

Inserting this value of f into (14) yields $r(p) = 2Cp$ and substituting $p = \frac{r}{2C}$ from the latter relation back to $f = Cp^2$ one finally finds $f = \frac{r^2}{4C}$. Using $R(g) = r(p)$ one arrives at $f(R) = \frac{R^2}{4C}$ for any real $C \neq 0$, i.e., the degenerate Lagrangian. In particular the potential cannot vanish identically. In fact, $V = 0$ implies $r(p) = f(r)/p$. Differentiating this relation with respect to r under assumption that $f'(r) \neq 0$ and $f''(r) \neq 0$ (the condition for $r(p)$ to exist) one arrives at $ff''/p^2 = 0$ implying $f'' = 0$. This contradiction shows that $V \not\equiv 0$.

For admissible Lagrangians the potential is variable and this feature will be used to establish stability.

Finally we comment on stability of a CGS solution in the presence of some matter. In our opinion it is the stability of pure gravity (only the metric and the scalar) that is crucial for physical viability of the theory while exotic forms of matter violating DEC can make the ground state unstable even in general relativity as it found in the two examples mentioned in section 2. Yet recently there appeared claims (see e.g. [36]) that the very presence matter (perfect fluid stars) renders $f(R)$ gravity unstable. We show now that this is not the case. The point is that the property of the variational matter energy-momentum tensor (the stress tensor for short) to satisfy DEC is preserved under a conformal map of the metric. If one assumes that Jordan frame is physical and minimally couples a given species of matter Ψ in this frame, then the stress tensor $t_{\mu\nu}(\Psi, g)$ satisfies DEC by assumption. The field equations (1) for the metric read then⁸

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \nabla_\mu \nabla_\nu f'(R) + g_{\mu\nu} \square f'(R) = \kappa^2 t_{\mu\nu}(\Psi, g). \quad (15)$$

The conformal map $\tilde{g}_{\mu\nu} = pg_{\mu\nu}$ makes the matter Lagrangian explicitly dependent on the scalar gravity p and the stress tensor in EF for Ψ alone cannot be unambiguously derived from it. It is therefore convenient to express the gravitational field equations in both the frames in terms of $t_{\mu\nu}$ which is already defined as the variational one in terms of the physical metric (i.e. in JF). The metric field equations in EF replacing (11) are then [34]

$$\tilde{G}_{\mu\nu}(\tilde{g}) = \kappa^2 T_{\mu\nu}(\phi, \tilde{g}) + \frac{\kappa^2}{p} t_{\mu\nu}(\Psi, \frac{1}{p}\tilde{g}). \quad (16)$$

⁸The choice of a constant coefficient in front of $f(R)$ in the Lagrangian should be determined by a Newtonian limit of the theory. This can be unambiguously done in the case where Minkowski space is the ground state. If the CGS solution under consideration is dS or AdS, the Newtonian limit is not well defined and the coefficient is undetermined. This trouble does not affect the present argument.

Since $p > 0$ in a vicinity of the CGS solution and DEC holds for both the stress tensors in EF⁹, it also holds for the total stress tensor $T_{\mu\nu} + \frac{1}{p}t_{\mu\nu}$. This means that matter cannot destroy stability of the ground state if it is stable in pure gravity theory. We comment on the instability found in [36] in section 7.

5 Positive energy theorem for anti-de Sitter space

We emphasize that the applied here method of proving stability of dS, AdS or \mathcal{M} spaces is based on the assumption that the scalar component of gravity satisfies in EF the dominant energy condition, what is equivalent to $V(\phi) \geq 0$. The fact that it implies positivity of total ADM or AD energy is not used. Nevertheless we shall consider for the moment this energy. In [34] we proved that if $L = f(R)$ is analytic at $R = 0$ and its expansion is $L = R + aR^2 + \dots$ and the potential $V(\phi)$ in EF is non-negative, the ADM energy of a spacetime which is asymptotically flat is the same in both Jordan and Einstein frames and is non-negative. Near \mathcal{M} the potential behaves as $V = \frac{1}{2}aR^2 + O(R^3)$, whence $V > 0$ for $a > 0$. An analogous positive-energy theorem may be proved in restricted NLG theories for spacetimes which are asymptotically AdS space. The case of spacetimes which asymptotically converge to de Sitter space is more complicated because dS is not globally static and we disregard it.

Let $\bar{g}_{\mu\nu}$ be the metric of AdS space in the following coordinates:

$$d\bar{s}^2 = \bar{g}_{\mu\nu}dx^\mu dx^\nu = -(1 + \frac{r^2}{a^2})dt^2 + (1 + \frac{r^2}{a^2})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \quad (17)$$

the cosmological constant is $\Lambda = -\frac{3}{a^2}$, $a = \text{const} > 0$ and $\bar{R} = \lambda = 4\Lambda$. Let $g_{\mu\nu}$ be a solution of the field equations (1) in JF which asymptotically approaches AdS metric (17), $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$. Clearly $g_{\mu\nu}$ is a solution to Einstein field equations $G_{\mu\nu}(g) = \theta_{\mu\nu}$ in HJF, then the Abbott–Deser approach [7] applies and the total energy of the fields $g_{\mu\nu}$ and p is given by their formula, which in the case of (17) reduces to

$$\begin{aligned} E_{AD}[g] = & \frac{c^4}{16\pi G} \lim_{r \rightarrow \infty} \int \sin\theta d\theta d\phi [-r^2 \partial_1 h_{00} + \frac{r^6}{a^4} \partial_1 h_{11} + \frac{r^2}{a^2} (\partial_2 h_{12} + \frac{1}{\sin^2\theta} \partial_3 h_{13}) \\ & + 3r h_{00} + \frac{3}{a^4} r^5 h_{11} - \frac{r}{a^2} (h_{22} + \frac{h_{33}}{\sin^2\theta}) + \frac{r^2}{a^2} h_{12} \text{ctg}\theta], \end{aligned} \quad (18)$$

here $x^i = (r, \theta, \phi)$ and the timelike Killing vector in the Abbott–Deser formula is chosen as $\xi^\mu = \delta_0^\mu$, then its normalization at $r = 0$ is $\xi^\mu \xi_\mu = -1$. In general all the components

⁹This frame invariance of DEC does not apply to the scalar gravity p since in any frame its stress tensor is defined as that equal in pure gravity (no matter) to the Einstein tensor of the given metric.

of $h_{\mu\nu}$ are algebraically independent and the requirement that separately each term in the integrand of (18) gives rise to a finite integral (what amounts to requiring that each term be independent of r) provides the asymptotic behaviour of:

h_{00} , h_{22} and h_{33} are of order r^{-1} , $h_{11} = O(r^{-5})$ and $h_{12} = O(r^{-2}) = h_{13}$.

A spacetime being asymptotically anti-de Sitter space is defined in [37] and according to this definition a solution approaches AdS slower than is required by finiteness of its energy. We assume that the six components of $h_{\mu\nu}$ behave as shown above while the remaining four components, which do not enter the energy integral, tend to AdS as in the definition in [37], $h_{01} = O(r^{-1})$ and h_{02} , h_{03} and h_{23} are $O(r)$. Under these assumptions the scalar $R(g)$ for a solution with finite energy approaches $\bar{R} = 4\Lambda$ as $R \rightarrow 4\Lambda + O(r^{-2})$.

In Einstein frame an analogous integral expression for $E_{AD}[\tilde{g}]$ holds for the corresponding solution $\tilde{g}_{\mu\nu}$ with $h_{\mu\nu}$ replaced by $\tilde{h}_{\mu\nu} = p(R)h_{\mu\nu}$. For $r \rightarrow \infty$ the conformal factor is $p = f'(R) = f'(\bar{R} + O(r^{-2})) = f'(4\Lambda) + O(r^{-2})$ (assuming that $f''(4\Lambda) \neq 0$ and finite), whence $E_{AD}[\tilde{g}] = f'(4\Lambda)E_{AD}[g]$ is finite. This energy is positive according to the positive energy theorem in general relativity provided $V(\phi) > 0$. Since $f'(4\Lambda) > 0$ by assumption, we get that in spite of the indefiniteness of the tensor $\theta_{\mu\nu}(g, p)$ in HJF the *positive-energy theorem for restricted NLG theories* holds:

(i) if $L = f(R)$ admits AdS space with $\bar{R} = 4\Lambda < 0$ as a solution, (ii) $f'(4\Lambda) > 0$ and $f''(4\Lambda) \neq 0$ is finite, (iii) the potential $V(\phi)$ in EF is non-negative and (iv) a solution $g_{\mu\nu}$ in JF or equivalently the pair $(g_{\mu\nu}, p)$ in HJF tends sufficiently quickly to AdS space for $r \rightarrow \infty$, then the total energy in JF is equal to the AD energy in HJF and positive and proportional to that in EF,

$$E_{AD}[g] = (f'(4\Lambda))^{-1} E_{AD}[\tilde{g}] > 0.$$

Recall that the AD definition of conserved energy only makes sense in HJF (and EF) since we have no notion of total energy for fourth-order equations of motion. Total gravitational energy in Jordan frame is therefore defined as a quantity equal to that in HJF.

6 Minimum of the potential and stability

In order to establish whether the potential for the scalar gravity ϕ in Einstein frame is non-negative in a vicinity of a candidate ground state solution \mathcal{M} , dS or AdS, it is necessary to calculate the first and second derivative of V at this state. To this end one first determines the derivative $\frac{dr}{dp}$ of the inverse function $R = r(p)$ to the definition of

the scalar, $p = \frac{df}{dR}$. It is equal to

$$\frac{dR}{dp} = \left(\frac{dp}{dr} \right)^{-1} \Big|_{r=r(p)} = \left[\frac{d^2 f}{dR^2} \Big|_{R=r(p)} \right]^{-1}. \quad (19)$$

Applying this outcome to the potential in (10) one finds

$$\frac{dV}{dp} = \frac{1}{2\kappa^2 p^2} \left[\frac{2}{p} f(r(p)) - r(p) \right] \quad (20)$$

and this expression should also be inserted into the field equation (12) for ϕ .

Consider a CGS solution in Jordan frame with $G_{\alpha\beta} = -\frac{1}{4}\lambda\bar{g}_{\alpha\beta}$ and $R(\bar{g}) = \lambda$ where λ is a solution to (2). In HJF the scalar p at this state is $p_0 \equiv p(\lambda) = f'(\lambda) > 0$. For the function $r(p)$ one has $r(p_0) = r(f'(\lambda)) = \lambda$. Under the conformal map from HJF to EF the metric $\bar{g}_{\mu\nu}$ of the CGS solution is mapped to $\tilde{g}_{\mu\nu} = p_0\bar{g}_{\mu\nu} = f'(\lambda)\bar{g}_{\mu\nu}$ and the scalar ϕ is equal to $\phi_0 = \sqrt{\frac{3}{2}}\frac{1}{\kappa} \ln f'(\lambda)$. The Einstein tensor remains invariant under a constant rescaling of the metric, hence

$$G_{\mu\nu}(\bar{g}) = \tilde{G}_{\mu\nu}(\tilde{g}) = -\frac{1}{4}\lambda\bar{g}_{\mu\nu} = -\frac{1}{4}\frac{\lambda}{f'(\lambda)}\tilde{g}_{\mu\nu}$$

and this allows one to define a cosmological constant in Einstein frame as

$$\Lambda \equiv \frac{\lambda}{4f'(\lambda)}.$$

Thus \mathcal{M} , dS and AdS spaces in JF (and HJF) are respectively mapped onto \mathcal{M} , dS and AdS spaces in EF satisfying $\tilde{G}_{\mu\nu}(\tilde{g}) = -\Lambda\tilde{g}_{\mu\nu}$ and being the CGS solutions in Einstein frame. Physical excitations of the field ϕ in EF should be counted from its ground value ϕ_0 , i.e., are equal $\psi \equiv \phi - \phi_0$, then $p = f'(\lambda) \exp(\sqrt{\frac{2}{3}}\kappa\psi)$. The potential V at $\phi = \phi_0$ is

$$V(\phi_0) = \frac{1}{2\kappa^2 p_0^2} [\lambda f'(\lambda) - f(\lambda)]$$

and applying (2) it equals to

$$V(\phi_0) = \frac{1}{4\kappa^2} \frac{\lambda}{f'(\lambda)} = \frac{\Lambda}{\kappa^2}.$$

The potential for the scalar excitation ψ is then

$$U(\psi) \equiv V(\phi) - V(\phi_0) = V(p(\phi)) - \frac{\Lambda}{\kappa^2} \quad (21)$$

and vanishes for vanishing excitation, $U(0) = 0$. The field equation (11) is now modified to (hereafter $\tilde{g}_{\mu\nu}$ denotes any dynamical metric in EF, not only the maximally symmetric CGS solutions)

$$\tilde{G}_{\mu\nu}(\tilde{g}) + \Lambda\tilde{g}_{\mu\nu} = \kappa^2 \left[\psi_{,\mu}\psi_{,\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{g}^{\alpha\beta}\psi_{,\alpha}\psi_{,\beta} - \tilde{g}_{\mu\nu}U(\psi) \right]. \quad (22)$$

The first derivative of U with respect to ψ (or p) vanishes when $\frac{2}{p}f(r(p)) - r(p) = 0$ and this equation viewed as an equation for r coincides with eq. (2). Hence $\frac{dU}{d\psi} = 0$ only at the CGS solutions with $r(p_i) = \lambda_i = r(f'(\lambda_i))$, $i = 1, \dots, n$. In other terms the equation $\lambda f'(\lambda) = 2f(\lambda)$ determines all stationary points of U . At each of these points the potential $U_i(\psi) = V(\phi) - \Lambda_i/\kappa^2$ vanishes provided $\Lambda_i = \lambda_i(4f'(\lambda_i))^{-1}$. On the other hand $U(\psi)$ (for a fixed value of λ) may also vanish at some points r_i different from the solutions λ_i but these are not its stationary points; if such points exist the dominant energy condition is broken and a kind of (nonlinear) instability may develop.

The second derivative of the potential, determining its behaviour at a stationary point is, from (20) and (19),

$$\frac{d^2U}{d\psi^2} = \frac{1}{3p^2} \left[-4f(r) + pr + \frac{p^2}{f''(r)} \right]. \quad (23)$$

At the stationary point $R = r = \lambda$, $\psi = 0$ and $p = f'(\lambda)$, then

$$\frac{d^2U}{d\psi^2} \Big|_{\psi=0} = \frac{-\lambda}{3f'(\lambda)} + \frac{1}{3f''(\lambda)}. \quad (24)$$

For regular Lagrangians we are considering in this work one has $f''(\lambda) \neq 0$ finite. The potential $U(\psi)$ is non-negative if it attains minimum at $\psi = 0$, i.e.

$$\frac{d^2U}{d\psi^2} \Big|_{\psi=0} > 0.$$

Stability conditions were also derived by other authors applying different methods. Our condition is equivalent to that found in [23] which after using $\lambda f'(\lambda) = 2f(\lambda)$ reads

$$3f'(\lambda) \frac{d^2U}{d\psi^2} \Big|_{\psi=0} \geq 0. \quad (25)$$

The linear perturbation method applied in [23] implies that stability occurs whenever the weak inequality in (25) holds. Cognola et al. [38] employ a minisuperspace approach to the stability problem (perturbations are spatially homogeneous) and get a stability condition of de Sitter space which is equivalent¹⁰ to ours; also computing one-loop

¹⁰The formula for the condition seems to be misprinted since it disagrees with their earlier work.

quantization corrections to $L = f(R)$ they find this condition for dS background [39]. Yet Song et al. [40, 41] define stability of spatially flat R–W spacetime in a nonstandard way: a gravity theory is stable if it approaches general relativity at high curvatures (for small R the theory should diverge from GR by definition); this cosmological criterion does not deal with a ground state solution.

The derivation of (24) holds both for $\lambda = 0$ and $\lambda \neq 0$. The case $\lambda = 0$ is simpler to study. In this case $f(0) = 0$ and assuming analyticity around $R = 0$ one has

$$f(R) = R + aR^2 + \sum_{n=3}^{\infty} a_n R^n, \quad (26)$$

then $f(0) = 0$, $f'(0) = 1$, $f''(0) = 2a \neq 0$ and $U''(0) = \frac{1}{6a}$. For $a > 0$ the potential $U \geq 0$ and the scalar field satisfies DEC. For spacetimes which are asymptotically flat it is known [34] that $E_{ADM}[g] = E_{ADM}[\tilde{g}, \psi] \geq 0$ and the total energy vanishes only in Minkowski spacetime, $\tilde{g}_{\mu\nu} = \eta_{\mu\nu} = g_{\mu\nu}$ and $\psi = 0$.

In de Sitter space ($\lambda > 0$) there are following cases:

- for $f''(\lambda) < 0$ the potential attains maximum at $\psi = 0$ and the space is unstable;
- for $f''(\lambda) > 0$ and $f'(\lambda) > \lambda f''(\lambda)$ there is minimum of U and $U(\psi) \geq 0$, hence the space is stable;

- for $f''(\lambda) > 0$ and $f'(\lambda) < \lambda f''(\lambda)$ one finds $U''(0) < 0$ and instabilities develop.

For anti-de Sitter space the situation is reversed:

- for $f''(\lambda) < 0$ and $f'(\lambda) > \lambda f''(\lambda)$ the negative potential attains maximum and the space is unstable;

- for $f''(\lambda) < 0$ and $f'(\lambda) < \lambda f''(\lambda)$ the potential is at minimum and AdS is stable;

- for $f''(\lambda) > 0$ the minimum of U shows stability of the space.

Finally we return to the problem of singular Lagrangians (13) for which $f(\lambda) = 0 = f'(\lambda)$; for them the derivative (24) is divergent and the method of deriving it does not work. One may instead apply the gauge invariant perturbation method for de Sitter space directly in Jordan frame which gives rise [23] to the inequality (25). Let the lowest nonvanishing coefficient in the series (13) be a_k . If $k > 2$ then also $f''(\lambda) = 0$ and the expression (25) becomes indeterminate. In order to give it a definite value we define a function

$$J(R) \equiv -R + \frac{f'(R)}{f''(R)} \quad (27)$$

and define $J(\lambda)$ as its limit for $R \rightarrow \lambda$. Let $R = \lambda + \epsilon$, $|\epsilon| \ll 1$, then $f'(R) = k a_k \epsilon^{k-1} + O(\epsilon^k)$, $f''(R) = k(k-1) a_k \epsilon^{k-2} + O(\epsilon^{k-1})$ and

$$J(\lambda + \epsilon) = -(\lambda + \epsilon) + \frac{\epsilon}{k-1} + O(\epsilon^2).$$

Hence the stability criterion is $J(\lambda) = -\lambda \geq 0$. Recall that the method works only in dS space, $\lambda \geq 0$, therefore the conclusion is that for all NLG theories having Lagrangians of the form (13) with $\lambda > 0$, de Sitter space (as a CGS solution¹¹) is unstable. None of the methods can be applied to these Lagrangians in the case $\lambda < 0$. It might be argued that by continuity the criterion $J(\lambda) \geq 0$ should also work for $\lambda < 0$, then all AdS spaces would be stable in these theories. However this argument is of little reliability.

In Paper I an astonishing theorem was mentioned to the effect that an anti-de Sitter space may be stable in spite of the fact that the scalar ψ has a tachyonic mass (i.e., the potential $U(\psi) < 0$ and attains maximum at this space) [42]. In fact, if small fluctuations of the scalar gravity vanish sufficiently fast at spatial infinity of AdS space (i.e., for $r \rightarrow \infty$ in the metric (17)), the kinetic energy of the field dominates over its negative potential energy and the total energy of the scalar,

$$E(\psi) = - \int d^3x \sqrt{-g} T^{0\nu} \xi_\nu,$$

where $T^{0\nu}$ is given in (22), is finite and positive, $0 < E(\psi) < \infty$. This occurs if $\frac{d^2U}{d\psi^2} > \frac{3}{4}\Lambda$ at $\psi = 0$. Since the energy of gravitational perturbations of AdS space is positive [7], the total energy of metric and scalar field fluctuations is positive and Breitenlohner and Freedman conclude [42] that AdS space is stable against these (small) fluctuations. Applying the definition of Λ in Einstein frame arising in NLG theories, the condition of stability of AdS in the case of maximum of the potential reads

$$0 > \frac{d^2U}{d\psi^2} \Big|_{\psi=0} > \frac{3\lambda}{16f'(\lambda)}. \quad (28)$$

It should be stressed, however, that in this case the DEC is violated (only the total energy of the scalar is positive). From the viewpoint of a rigorous mathematical approach to the stability problem the condition (28) is rather unreliable [20].

7 Examples: specific Lagrangians

We now apply the stability criteria of the previous section to a number of Lagrangians, some of which were already discussed in the literature. We assume that the Lagrangians depend on one dimensional constant μ and some dimensionless constants. μ is positive and has dimension $(\text{length})^{-1}$ so that R/μ^2 is a pure number.

$$1. \quad L = R + \frac{\mu^{4n+4}}{R^{2n+1}}, \quad n = 0, 1, \dots \quad (29)$$

¹¹Besides $R = \lambda$ there are in general other solutions to $Rf'(R) = 2f(R)$, e.g. for $f(R) = a(R - \lambda)^3$ the other solution is $R = -2\lambda$.

This Lagrangian belongs to the class which admits no CGS solutions since it is given by eq. (3) for $a = 0$ and $F(R) = -R - (2n + 3)\mu^{4n+4}R^{-(2n+1)}$. Clearly it should be rejected. Yet according to Sawicki and Hu [41] the theory for $n = 0$ converges to general relativity for large R and in this sense is admissible.

$$2. \quad L = R + \frac{\mu^{4n+2}}{R^{2n}}, \quad n = 1, 2, \dots \quad (30)$$

There is only one CGS solution with $\lambda = \lambda_- \equiv -(2n + 2)^{\frac{1}{2n+1}}\mu^2 < 0$, $f'(\lambda) = (2n + 1)(n + 1)^{-1} > 0$ and the scalar is

$$p(r) = 1 - 2n \left(\frac{\mu^2}{r} \right)^{2n+1}. \quad (31)$$

We consider spacetimes with $R = r$ in vicinity of $R = \lambda_-$, so that $-\infty < r < 0$ and $1 < p < +\infty$. The inverse function and the potential are respectively

$$r = - \left(\frac{2n}{p-1} \right)^{\frac{1}{2n+1}} \mu^2, \quad U = U_- = - \frac{(2n+1)\mu^2}{2\kappa^2 p^2} \left(\frac{p-1}{2n} \right)^{\frac{2n}{2n+1}} - \frac{\Lambda_-}{\kappa^2} \quad (32)$$

with $\Lambda_- < 0$. The potential is always non-negative and $U \leq \left| \frac{\Lambda_-}{\kappa^2} \right|$. It attains minimum at $\psi = 0$ showing that AdS space is a stable ground state solution for this theory.

$$3. \quad L = R + \frac{\mu^{2n+2}}{R^n} \quad \text{for} \quad -1 < n < 0 \quad \text{real}, \quad (33)$$

for non-integer n the function R^n is replaced by $|R|^n$. The background evolution of the R-W spacetime is cosmologically acceptable [2] and solutions of the linear perturbation equations for this Lagrangian are not incompatible with the observational data [43]. However the equation for a ground solution gives rise to the contradiction $|R|^{n+1} = -(n+2)\mu^{2n+2}$ implying that the theory is untenable.

$$4. \quad L = R - \frac{\mu^{4n+2}}{R^{2n}}, \quad n = 1, 2, \dots \quad (34)$$

Here λ and $r(p)$ have the same moduli and opposite sign to those in the case 2: dS space is the unique CGS solution for $\lambda = \lambda_+ = -\lambda_- > 0$ and $r(p) > 0$. Accordingly, $\Lambda = \Lambda_+ = -\Lambda_-$. Now we take r around $r = \lambda_+$, and again $1 < p < \infty$. The potential is $U = U_+ = -U_-$, hence it is contained in the interval $-\frac{\Lambda_+}{\kappa^2} \leq U_+ \leq 0$. This indicates that U has maximum at $\psi = 0$ and this fact is confirmed by a direct computation. In conclusion, de Sitter space is unstable and this theory is discarded as unphysical.

$$5. \quad L = R - \frac{\mu^{4n+4}}{R^{2n+1}}, \quad n = 0, 1, \dots \quad (35)$$

This Lagrangian has been most frequently studied in applications to the accelerating universe, usually for $n = 0$. Most expressions here are akin to respective ones in the case 4. The field p is always greater than 1 and there are two CGS solutions for $\lambda_{\pm} = \pm(2n+3)^{\frac{1}{2n+2}}\mu^2$, hence $p(\lambda_{\pm}) = p(\lambda_{-}) = \frac{4n+4}{2n+3}$. The two CGS solutions define two different sectors of the theory which should be separately studied.

A. De Sitter space sector.

$\lambda = \lambda_{+} > 0$ and the sector comprises all positive values of r . The inverse function is

$$r(p) = r_{+}(p) = \left(\frac{p-1}{2n+1}\right)^{\frac{-1}{2n+2}} \mu^2 \quad (36)$$

giving rise to the potential [44]

$$U(p(\psi)) = U_{+} = \frac{n+1}{\kappa^2 p^2} \left(\frac{p-1}{2n+1}\right)^{\frac{2n+1}{2n+2}} \mu^2 - \frac{\Lambda_{+}}{\kappa^2} \quad (37)$$

which is always non-positive and attains maximum at dS space. This space is then unstable (for $n = 0$ it was found in [44, 23, 39]) and this sector of the theory must be rejected (on other grounds this conclusion was derived in [45]).

Seifert [36] finds that gravity theory (35) is highly unstable in the presence of matter: a static spherically symmetric solution becomes unstable to linear spherically symmetric perturbations if perfect fluid matter forms a quasi-Newtonian polytropic star. This result is derived applying an intricate variational method and requires very long computations. We note that (besides the fact that the Newtonian limit is not well defined there) the author assumes that R is approximately equal to the stellar matter density. This means that he deals with spherically symmetric perturbations of dS space. Since this space is unstable in pure gravity (35) it would be rather surprising if a small amount of matter could stabilize it.

B. Anti-de Sitter space sector.

Its existence (for $n = 0$) was first noticed in [44], then in [46], but its properties were never analyzed in detail, probably due to the fact that a negative Λ does not fit the observed accelerated expansion. $\lambda = \lambda_{-} < 0$ and accordingly $-\infty < r < 0$, hence $r(p) = r_{-}(p) = -r_{+}(p)$ and $U = U_{-} = -U_{+}$ with $\Lambda_{-} = -\Lambda_{+}$. This potential is non-negative and has minimum at $\psi = 0$. This sector has a stable ground state solution¹² and in this sense it forms a viable gravity theory. The scalar gravity has mass being

¹²In [47] it is claimed that Lagrangians given in cases 4 and 5 (for both n even and odd) always develop instabilities while Lagrangians in cases 1 and 2 always describe a stable theory.

a function of n , for $n = 0$ it is $m^2 = \frac{3\sqrt{3}}{4}\mu^2$ while for $n \rightarrow \infty$ it tends to $m^2 \rightarrow \frac{\mu^2}{6}$. Disregarding the incompatibility of this theory with the cosmic acceleration, one may make a rough estimate of μ . Since Λ is of order $-\mu^2$ for all $n \geq 0$ and the observational limit is $|\Lambda| \leq 10^{-52}\text{m}^{-2}$ one gets an upper limit $\mu \leq 10^{-26}\text{m}^{-1}$ or $\mu\hbar c \leq 10^{-33}\text{eV}$, very small indeed.

$$6. \quad L = R \left(\ln \frac{R}{\mu^2} \right)^q, \quad q > 0. \quad (38)$$

One assumes $R > 0$. A unique possible ground state is dS space with $\lambda = e^q \mu^2$, then $p(\lambda) = 2 \exp(q \ln q)$ and $f''(\lambda) = \mu^{-2}(2q-1) \exp[(q-1) \ln q - q]$. The Lagrangian must be regular, i.e. $f''(\lambda) \neq 0$ implying $q \neq 1/2$. From (24)

$$U''(0) = \frac{\mu^2}{6(2q-1)} e^{q(1-\ln q)}$$

hence for $0 < q < 1/2$ dS space is unstable while for $q > 1/2$ de Sitter space is a stable ground state. The function $p(r)$ may be inverted and then the potential can be explicitly calculated only for $q = 1$ or 2 . According to [2] this theory is cosmologically acceptable for any $q > 0$ though the matter era begins too early and its duration is too long.

$$7. \quad L = \alpha R - \frac{\mu^2}{\sinh \frac{R}{\mu^2}}, \quad \alpha \geq 0. \quad (39)$$

This Lagrangian appeared in the metric-affine approach to gravity [48]. The equation $Rf'(R) - 2f(R) = 0$ cannot be analytically solved even in the case $\alpha = 0$ (it can only be shown that the roots do not lie close to $R = 0$) and for practical reasons this theory must be rejected.

$$8. \quad L = \mu^2 \left(\ln \frac{R}{\mu^2} + \frac{1}{2} \right) + \frac{a}{\mu^2} R^2, \quad a > 0. \quad (40)$$

One may start from a more general Lagrangian [49]

$$L = \gamma R + b \ln(cR) + \frac{a'}{\mu^2} R^2, \quad a', b, c > 0,$$

but then eq. (2) for λ cannot be solved analytically. We therefore set $\gamma = 0$ and multiply L by μ^2/b and define $a = \frac{a'}{b}\mu^2$; finally we choose such value of c as to get a simple expression for λ . A unique solution to (2) is then $\lambda = \mu^2$ and

$$p = \frac{\mu^2}{r} + \frac{2a}{\mu^2} r, \quad (41)$$

$r > 0$. To invert this function we first notice that $p(r) \rightarrow \infty$ for both $r \rightarrow 0$ and $r \rightarrow \infty$ and has minimum at $r_0 = \mu^2/\sqrt{2a}$ equal to $p(r_0) = 2\sqrt{2a}$. Hence $p(r)$ may be inverted either in the interval $0 < r < r_0$ or $r > r_0$. To choose the correct interval one must establish whether $\lambda = \mu^2$ belongs to the ascending or descending branch of $p(r)$ and this depends on the value of a . We assume $a > 1/2$, then $\mu^2 > r_0$ and dS space lies on the ascending branch of p (for $a < 1/2$ a similar procedure can be performed). Solving (41) one chooses the larger root (both roots are positive),

$$r(p) = \frac{\mu^2}{4a} \left(p + \sqrt{p^2 - 8a} \right) \quad (42)$$

since $r \rightarrow \infty$ corresponds to $p \rightarrow \infty$. The potential is

$$U = \frac{1}{16a\kappa^2 p} \left[P(p) - \frac{8a}{p} (\ln P - \ln(4a)) \right] \mu^2 - \frac{\Lambda}{\kappa^2}, \quad (43)$$

where $P \equiv p + \sqrt{p^2 - 8a}$ and $\Lambda = \frac{\mu^2}{4(2a+1)}$. This implies $f'(\lambda) = 2a+1 > \lambda f''(\lambda) = 2a-1$ and the potential has minimum at $\psi = 0$. This theory has dS space as a stable ground state and is viable.

The case $a = 1/2$ is singular since $f''(\mu^2) = 0$ and $p(r)$ cannot be inverted around $r = \mu^2$ while $f'(\mu^2) = 2$. Formally the conformal map to EF exists at this point but the potential U cannot be defined there. None of the methods to check the stability does work there and it is reasonable to disregard this case.

9. The limiting case $a = 0$ of the Lagrangian (40) requires a separate treatment. Again $\lambda = \mu^2$, $p = \frac{\mu^2}{r}$, and $r(p) = \mu^2/p > 0$. $f'(\lambda) = 1$ and $f''(\lambda) = -\frac{1}{\mu^2}$ give rise to $U''(0) = -\frac{2}{3}\mu^2$. De Sitter space is unstable making the theory untenable.

The additive constant appearing in this Lagrangian (as well as in the case 8) is inessential in the sense that it only affects the absolute value of λ (but not its sign) and has no influence on stability properties of dS space. In fact, for a Lagrangian

$$L = \mu^2 \left(\ln \frac{R}{\mu^2} + a \right),$$

a real dimensionless, one gets again $p = \frac{\mu^2}{r}$ and the value of λ is shifted to $\lambda = \mu^2 \exp(\frac{1}{2} - a)$, hence it is still dS space. Then $U''(0) = -\frac{\mu^2}{3} [1 + \exp(1 - 2a)]$ implying instability of the space for any a . This case is, however, exceptional: we will see below that in general not only μ but also dimensionless parameters in L determine stability of CGS solutions.

$$10. \quad L = \mu^2 \left(\frac{R}{\mu^2} \right)^\alpha \quad (44)$$

for α rational (negative and positive) has also attracted some attention [23, 48, 50] since it is a scale-invariant theory. For non-integer α one takes $|R|^\alpha$. If $\alpha < 0$ the equation $Rf'(R) = 2f(R)$ is solved only by $R = \pm\infty$ and we reject this case. For $\alpha = 2$ one gets the degenerate Lagrangian R^2 which we disregard. For $\alpha > 2$ integer this is a singular Lagrangian (13) discussed in section 4 having $\lambda = 0$ and the criterion $-\lambda \geq 0$ yields that Minkowski space is stable for these theories. Putting aside the obvious case $\alpha = 1$ one considers $\alpha > 0$ non-integer. $f(0) = 0$ always. For $0 < \alpha < 1$ both $f'(0)$ and $f''(0)$ are infinite, for $1 < \alpha < 2$ there is $f'(0) = 0$ and $f''(0) = \infty$ and for $\alpha > 2$ both $f'(0) = f''(0) = 0$. Once again one may apply the function $J(R)$ defined in (27) and it is equal $J = \frac{2-\alpha}{\alpha-1}R$ so that $J(0) = 0$ and for all three cases the criterion $J(0) \geq 0$ is satisfied. One may thus claim that for all $\alpha > 0$ Minkowski space is the unique stable ground state, nevertheless it is difficult to avoid impression that for $\alpha \neq 1$ the theory is bizarre and rather unphysical (and furthermore in conflict with the astronomical observations, as mentioned in Paper I).

$$11. \quad L = R - \frac{\mu^4}{R} + \frac{a}{\mu^2}R^2, \quad (45)$$

a real [25, 24, 23]. There are two CGS solutions with $\lambda_\pm = \pm\sqrt{3}\mu^2$, which are the same as for the case $a = 0$ (Lagrangian (35) for $n = 0$) since the R^2 term does not contribute to λ . The attempt to find the inverse function $r(p)$ leads to a cubic equation and solving it would be impractical. We therefore quit from computing the explicit form of the potential (an implicit form of V is given in [25]) and restrict ourselves to studying its extrema.

A. De Sitter sector for $\lambda = \lambda_+$.

The condition $p(\lambda_+) > 0$ requires $a > -\frac{2}{3\sqrt{3}}$. This condition does not determine the sign of

$$f''(\lambda_+) = \frac{2}{\mu^2} \left(\frac{-1}{3\sqrt{3}} + a \right)$$

and from (24) one finds:

- for $-\frac{2}{3\sqrt{3}} < a < \frac{1}{3\sqrt{3}}$ dS space is unstable and
- for $a > \frac{1}{3\sqrt{3}}$ dS space is stable. Yet cosmologically the theory in this case is unacceptable since there is no standard matter era preceding the acceleration era [2].

We omit the singular case $a = \frac{1}{3\sqrt{3}}$ where $f''(\lambda_+) = 0$.

B. Anti-de Sitter sector with $\lambda = \lambda_-$.

Now the condition $p(\lambda_-) > 0$ requires $a < \frac{2}{3\sqrt{3}}$. From (24) one gets that for $-\frac{1}{3\sqrt{3}} < a < \frac{2}{3\sqrt{3}}$ the potential has minimum at $\psi = 0$ and AdS space is stable. Yet for $a < -\frac{1}{3\sqrt{3}}$ the potential has maximum. This, however, does not automatically imply the instability since one should furthermore apply the criterion (28) of positivity of scalar field energy. It follows from it that AdS space is

- stable for $-\frac{19}{9\sqrt{3}} < a < -\frac{1}{3\sqrt{3}}$ with respect to scalar field perturbations with positive energy,
- unstable for $a < -\frac{19}{9\sqrt{3}}$.

In the range of values of a for which the theory is stable in the standard sense (the potential has minimum) the mass of the scalar gravity excitations above dS space is

$$m_+^2 = 3\sqrt{3}[2(3\sqrt{3}a + 2)(3\sqrt{3}a - 1)]^{-1}\mu^2,$$

while in the case of AdS ground state it is

$$m_-^2 = 3\sqrt{3}[2(3\sqrt{3}a + 1)(2 - 3\sqrt{3}a)]^{-1}\mu^2.$$

The particle masses tend to infinity when a approaches the finite limits of the admissible range. m_+ monotonically decreases and becomes very small for large values of a while in the AdS sector the scalar particle mass attains minimum $m_-^2 = \frac{2}{\sqrt{3}}\mu^2$ at $a = (6\sqrt{3})^{-1}$.

In the interval $\frac{1}{3\sqrt{3}} < a < \frac{2}{3\sqrt{3}}$ the theory has two viable sectors: one with dS space ground state for $\Lambda_+ = 3\sqrt{3}[8(3\sqrt{3}a + 2)]^{-1}\mu^2$ and the other having AdS as a ground state with $\Lambda_- = -3\sqrt{3}[8(2 - 3\sqrt{3}a)]^{-1}\mu^2$. Classically these are two different physical theories, each with a unique ground state. One cannot claim that this is one theory having two different and distant (in the space of solutions) local minima of energy. Energetically these two states are incomparable, each of them has vanishing energy (defined with respect to itself) and assuming that one of these minima is lower than the other is meaningless [5]. One may only compare the masses of the scalar gravity in the two theories. The mass ratio $\left(\frac{m_+}{m_-}\right)^2$ decreases monotonically from infinity for a approaching $(3\sqrt{3})^{-1}$ to zero for a tending to $2(3\sqrt{3})^{-1}$. If one believes that this Lagrangian describes the physical reality a difficult problem arises: how does the nature choose which of the two theories with the same Lagrangian is to be realized? In our opinion the nature avoids this problem merely by avoiding this Lagrangian (and other ones with the same feature).

This Lagrangian illustrates a general rule: all the parameters appearing in a Lagrangian do contribute to determination of stable sectors (i.e., physically distinct theories) corresponding to it.

$$12. \quad L = R \exp\left(\frac{\theta\mu^2}{R}\right), \quad \theta = \pm 1. \quad (46)$$

To each value of θ there is one CGS solution with $\lambda = -\theta\mu^2$ and $p(\lambda) = 2/e > 0$, then $U''(0) = -\frac{1}{6}\theta e\mu^2$. For $\theta = -1$ the potential attains minimum and de Sitter space is classically stable. For $\theta = +1$ one applies the stability criterion (28) for AdS space and one gets that this solution is unstable according to this condition too. The Lagrangian (46) for $\theta = -1$ is also cosmologically preferred since it is asymptotically equivalent to

the Λ CDM model [2]. Unfortunately the function $p(r)$ cannot be inverted analytically and the explicit form of the potential is unavailable.

13. Finally we consider a class of "toy models" possessing infinite number of ground states. For convenience we introduce a dimensionless variable $x = R/\mu^2$ and assume

$$L = f(R) = \mu^2 F(x) = \mu^2 e^{2I(x)} \quad (47)$$

where

$$I(x) \equiv \int \frac{dx}{x + h(x)} \quad (48)$$

and $h(x)$ is a continuous periodic function taking both positive and negative values, $M_1 \leq h(x) \leq M_2$ with $M_1 < 0$ and $M_2 > 0$. The scalar field is

$$p = \frac{dF}{dx} = \frac{2}{x + h(x)} F(x)$$

and is positive if $x + h(x) > 0$. For an arbitrary h one cannot find $r(p)$ and the potential; here it is sufficient to determine CGS solutions and U'' at these states. The eq. (2) takes now the form $x \frac{dF}{dx} = 2F$ and since $F > 0$ it is equivalent to

$$x = \frac{2}{\frac{d}{dx} \ln F}. \quad (49)$$

On the other hand from the definitions (47) and (48) it follows that

$$\frac{2}{\frac{d}{dx} \ln F} = x + h(x), \quad (50)$$

hence those x which are solutions of (49) must also be solutions to $h(x) = 0$. Since $M_1 \leq h(x) \leq M_2$ there is at least one root of $h(x) = 0$ and for a continuous periodic function there is infinite number of zeros, $h(x_n) = 0$, $n = 0, 1, \dots$ and $\lambda_n = \mu^2 x_n$. Note that $x_n \neq 0$ since $\lambda = 0$ implies $f(0) = \mu^2 \exp(2I(0)) = 0$ while $I(0)$ is finite by its definition. The function $x + h(x)$ tends to $\pm\infty$ for $x \rightarrow \pm\infty$, hence there is a point $x = y$ such that $y + h(y) = 0$ and $y \neq 0$. To ensure that $x + h(x) > 0$ for $x > y$ one requires $x + h(x)$ be monotonic, i.e. $1 + h'(x) > 0$. Then $I(x)$ is defined (and positive) for all $x > y$. Denoting $I_n \equiv I(x_n)$ one finds that U'' at a point $R = \lambda_n$ is

$$\left. \frac{d^2 U}{d\psi^2} \right|_{\lambda_n} = \frac{\mu^2}{6} x_n^2 e^{-2I_n} \frac{h'(x_n)}{1 - h'(x_n)}. \quad (51)$$

The condition $h'(x) > -1$ does not determine the sign of the fraction and to this aim one must specify h . Here we choose as an example $h(x) \equiv \frac{1}{2}(\sin x - \cos x)$. Clearly

$h'(x) = \frac{1}{2}(\sin x + \cos x) > -1$ and the unique solution of $x + \frac{1}{2}(\sin x - \cos x) = 0$ is $y = 0,3183\dots$. The zeros $x_n > y$ of h are solutions to $\operatorname{tg} x = 1$ ($\cos x \neq 0$) and these are $x_n = \frac{\pi}{4} + n\pi$, $n = 0, 1, \dots$. At these points $h'(x_n) = (-1)^n \frac{\sqrt{2}}{2}$ and for n odd there is

$$\left. \frac{d^2 U}{d\psi^2} \right|_{\lambda_{2n+1}} < 0,$$

therefore the infinite sequence of dS spaces with curvatures $\lambda_{2n+1} = \mu^2 x_{2n+1}$ defines unphysical (unstable) sectors of the theory. Yet the other sequence for n even consists of dS spaces having curvatures $\lambda_{2n} = (2n + \frac{1}{4})\pi\mu^2$ which are stable for this Lagrangian. The scalar particles corresponding to these sectors have masses

$$m_{2n}^2 = \frac{(2n + \frac{1}{4})^2}{6(\sqrt{2} - 1)} \pi^2 \mu^2 e^{-2I_{2n}}.$$

8 Conclusions

In this paper we have investigated stability of ground state solutions in $L = f(R)$ gravity theories being either Minkowski, de Sitter or anti-de Sitter spaces. Stability may be studied in any frame and Einstein frame is particularly suitable to this aim since one may apply there the methods developed in general relativity. We have given an explicit, effective and simple method of checking stability of these spaces based on the dominant energy condition applied to the scalar component of the gravitational doublet. After applying the method to thirteen specific Lagrangians (their ground states are de Sitter and/or anti-de Sitter spaces) corresponding to 20 different cases (depending on values of parameters in L) it was found that, as it was a priori expected, half of them give rise to viable theories (9 viable versus 11 untenable ones). And a generic feature is the existence of multiple vacua (stable ground states), each generating a separate physical sector or rather a separate gravity theory, all having the same Lagrangian. Hence it is expected that there is an infinity of viable gravity theories. What to do with such a wealth of theories (all differing from each other only by the form of the potential for the scalar gravity field)?

We stress that it is incorrect merely to search for a theory which easily and immediately accounts for the big problem of cosmology—the acceleration of the universe. After all general relativity was not formulated to solve some urgent problems in celestial mechanics (the perihelion shift of Mercury) or in cosmology (non-existence of Newtonian cosmology) and for many years its confirmation was quite marginal. At the time of its advent its advantage was that it was physically much deeper and more general than Newton's gravity. And the same should be expected about a modified gravity which may ultimately replace Einstein's theory. Its physical content will be more relevant than

immediate observational confirmation.

Before a deep creative physical idea will appear we need further viability criteria to maximally reduce the set of viable gravity theories. Undoubtedly one of the most important ones will be the condition that a tenable theory must be in agreement with the Newtonian and post-Newtonian approximations to gravity—as soon as these approximations will be rigorously defined in de Sitter and anti-de Sitter backgrounds. One should, however, expect that the selected set will still be large and possibly infinite.

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References

- [1] L. M. Sokołowski, *Metric gravity theories and cosmology. I. Physical interpretation and viability*, Class. Quantum Grav. **24** (2007) 3391 [gr-qc/0702097].
- [2] L. Amendola, R. Gannouji, D. Polarski and S. Tsujikawa, Phys. Rev. **D75**: 083504, 2007 [gr-qc/0612180].
- [3] K. A. Bronnikov and S. G. Rubin, Phys. Rev. **D73** (2006) 124019; K. A. Bronnikov, R. V. Konoplich and S. G. Rubin, Class. Quantum Grav. **24** (2007) 1261 [gr-qc/0610003].
- [4] E. d’Hoker and R. Jackiw, Phys. Rev. **D26** (1982) 3517.
- [5] E. Witten, Nucl. Phys. **B195** (1982) 481.
- [6] R. P. Woodard, *Proceedings of 3rd Aegean Summer School "The Invisible Universe: Dark Matter and Dark Energy", September 2005*, [astro-ph/0601672].
- [7] L. F. Abbott and S. Deser, Nucl. Phys. **B195** (1982) 76.
- [8] G. W. Gibbons, C. M. Hull and N. P. Warner, Nucl. Phys. **B218** (1983) 173.
- [9] G. T. Horowitz and R. C. Myers, Phys. Rev. **D59** (1998) 026005.
- [10] P. Ginsparg and M. J. Perry, Nucl. Phys. **B222** (1983) 245.

- [11] D. Christodoulou and S. Klainerman, *The global nonlinear stability of Minkowski space*, Princeton Univ. Press, Princeton 1993; S. Klainerman and F. Nicolo, *The evolution problem in general relativity*, Birkhäuser Verlag, Boston 2003.
- [12] N. Zipser, *The global nonlinear stability of the trivial solution of Einstein–Maxwell equations*, Ph. D. thesis, Harvard University 2000, unpublished.
- [13] H. Lindblad and I. Rodnianski, [math.AP/0411109].
- [14] H. Friedrich, J. Geom. Phys. **3** (1986) 101; Commun. Math. Phys. **107** (1986) 587.
- [15] M. T. Anderson, Ann. H. Poincare **6** (2005) 801.
- [16] H. Friedrich, J. Diff. Geom. **34** (1991) 275.
- [17] J. M. Heinzle and A. D. Rendall, Commun. Math. Phys. **269** (2007) 1, [gr-qc/0506134].
- [18] A. Ishibashi and R. M. Wald, Class. Quantum Grav. **21** (2004) 2981.
- [19] H. Friedrich, J. Geom. Phys. **17** (1995) 125.
- [20] M. T. Anderson (2006), private communication.
- [21] G. T. Horowitz, Phys. Rev. **D21** (1980) 1445.
- [22] J. Traschen and C. T. Hill, Phys. Rev. **D33** (1986) 3519.
- [23] V. Faraoni, Phys. Rev. **D72** (2005) 061501; V. Faraoni and S. Nadeau, Phys. Rev. **D72** (2005) 124005.
- [24] R. Dick, Gen. Rel. Grav. **36** (2004) 217.
- [25] S. Nojiri and S. D. Odintsov, Phys. Rev. **D68** (2003) 123512.
- [26] J. D. Barrow and A. C. Ottewill, J. Phys. **A16** (1983) 2757.
- [27] A. Hindawi, B. A. Ovrut and D. Waldram, Phys. Rev. **D53** (1996) 5597.
- [28] A. D. Dolgov and M. Kawasaki, Phys. Lett. **B573** (2003) 1.
- [29] T. P. Sotiriou, Phys. Lett. **B645** (2007) 389.
- [30] G. V. Bicknell, J. Phys. **A7** (1974) 1061.
- [31] T. Clifton and J. D. Barrow, Phys. Rev. **D72** (2005) 123003.
- [32] G. Magnano, M. Ferraris and M. Francaviglia, Gen. Rel. Grav. **19** (1987) 465.

- [33] A. Jakubiec and J. Kijowski, Phys. Rev. **D37** (1988) 1406; Gen. Rel. Grav. **19** (1987) 719.
- [34] G. Magnano and L. M. Sokołowski, Phys. Rev. **D50** (1994) 5039, [gr-qc/9312008].
- [35] I. P. C. Heard and D. Wands, Class. Quantum Grav. **19** (2002) 5435, [gr-qc/0206085].
- [36] M. D. Seifert, [gr-qc/0703060].
- [37] W. Boucher, G. W. Gibbons and G. T. Horowitz, Phys. Rev. **D30** (1984) 2447.
- [38] G. Cognola, M. Gastaldi and S. Zerbini, [gr-qc/0701138].
- [39] G. Cognola and S. Zerbini, J. Phys. **A39** (2006) 6245.
- [40] Y.-S. Song, W. Hu and I. Sawicki, Phys. Rev. **D75** (2007) 044004 [astro-ph/0610532].
- [41] I. Sawicki and W. Hu, Phys. Rev. **D75** (2007) 127502 [astro-ph/0702278].
- [42] P. Breitenlohner and D. Z. Freedman, Phys. Lett. **115B** (1982) 197; Ann. Phys. (N.Y.) **144** (1982) 249.
- [43] B. Li and J. D. Barrow, Phys. Rev. **D75**: 084010, 2007 [gr-qc/0701111].
- [44] S. M. Carroll, V. Duvvuri, M. Trodden and M. S. Turner, Phys. Rev. **D70** (2004) 043528.
- [45] M. E. Soussa and R. P. Woodard, Gen. Rel. Grav. **36** (2004) 855.
- [46] G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov and S. Zerbini, JCAP **0502** (2005) 010 [hep-th/0501096].
- [47] V. Faraoni, Phys. Rev. **D74** (2006) 104017.
- [48] G. Allemandi, A. Borowiec and M. Francaviglia, Phys. Rev. **D70** (2004) 043524.
- [49] S. Nojiri and S. D. Odintsov, Gen. Rel. Grav. **36** (2004) 1765.
- [50] T. Clifton and J. D. Barrow, Phys. Rev. **D72** (2005) 103005.